Solution TMA 4295 Autumn 2020

Problem 1

a)

$$f(x_1,\dots,x_m|\theta) = \prod_{i=1}^{m} \theta^{x_i}(1-\theta)^{1-x_i} \underset{x_i\in\{0,1\}}{I}(x_i) = \theta^{\sum_1^m x_i}\cdot(1-\theta)^{m-\sum_1^m x_i}\cdot \prod_{i=1}^{m} \underset{x_i\in\{0,1\}}{I}(x_i)$$

Introducing $g(\sum_1^m x_i|\theta) = \theta^{\sum_1^m x_i}\cdot(1-\theta)^{m-\sum x_i}$ and $h(x_1,\dots,x_m) = \prod_{i=1}^{m} \underset{x_i\in\{0,1\}}{I}(x_i)$

we get $f(x_1,\dots,x_m|\theta) = g(\sum_1^m x_i|\theta)\cdot h(x_1,\dots,x_m)$ which shows that

$Y = \sum_{i=1}^{m} X_i$ is a sufficient statistic for θ.

b) Y is a sum of m independent and identically distributed Bernoulli (θ) variable $\Rightarrow$ $Y \sim B(m,\theta)$.

$$L(\theta|x_1,\dots,x_m) = \theta^{\sum_1^m x_i}(1-\theta)^{m-\sum_1^m x_i}\cdot \prod_{i=1}^{m} \underset{x_i\in\{0,1\}}{I}(x_i)$$

$$\Rightarrow \ln L(\theta|x_1,\dots,x_m) = \sum_{i=1}^{m} x_i \ln\theta + (m-\sum_1^m x_i)\ln(1-\theta) + \ln \prod_{i=1}^{m} \underset{x_i\in\{0,1\}}{I}(x_i)$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{\sum x_i}{\theta} - \frac{m}{1-\theta} + \frac{\sum x_i}{1-\theta} = 0 \iff \sum x_i - \theta\sum x_i - m\theta + \theta\sum x_i = 0$$

$$\iff \theta = \frac{\sum x_i}{m} \Rightarrow \text{MLE for } \theta,\ \hat\theta = \frac{\sum X_i}{m} = \frac{Y}{m}$$

$$\left.\frac{\partial^2 \ln L}{\partial\theta^2}\right|_{\theta=\frac{\sum x_i}{m}} = \frac{-m}{\theta(1-\theta)} < 0 \Rightarrow \text{maksimum}.$$

c) $\hat\delta(\underline{X}_m) = \hat\theta(1-\hat\theta) = \frac{Y}{m}\left(1-\frac{Y}{m}\right) = \frac{\sum X_i}{m}\left(1-\frac{\sum X_i}{m}\right)$

$$E[\hat\delta(\underline{X}_m)] = \frac{E[Y]}{m} - \frac{E[Y^2]}{m^2} = \frac{m\theta}{m} - \frac{(m\theta(1-\theta)+m^2\theta^2)}{m^2} = \frac{m-1}{m}\theta(1-\theta)$$

$$\delta^{*}(\underline{X}_m) = \frac{m}{m-1}\delta(\underline{X}_m)$$

d) $$M_X(t) = E[e^{tX}] = \sum_{x=0}^{1} e^{tx} P(X=x) = 1-\theta+\theta e^t$$

$$M_Y(t) = \prod_{i=1}^{m} M_{X_i}(t) = (1-\theta+\theta e^t)^m$$

$$Var[\delta^\circ(\underline{X}_m)] = \frac{m^2}{(m-1)^2}\left[E\left[\left(\frac{Y}{m}\left(1-\frac{Y}{m}\right)\right)^2\right] - \frac{(m-1)^2}{m^2}\theta^2(1-\theta^2)\right]$$

$$= \frac{m^2}{(m-1)^2}\left[\frac{E[Y^2]}{m^2} - \frac{2E[Y^3]}{m^3} + \frac{E[Y^4]}{m^4}\right] - \theta^2(1-\theta^2)$$

$$E[Y^3] = M_Y'''(0), \quad E[Y^4] = M_Y^{iv}(0)$$

e) The binomial distribution is an exponential family and for $p \in (0,1)$, the family is complete. Y is a complete statistic for θ and $\delta^\circ(\underline{X}_m)$ is a function $\phi(Y)$ of Y, which is unbiased for $\theta(1-\theta)$. $\delta^*(\underline{X}_m)$ is therefore the unique UMVU estimator for $\delta(\theta)$

$$f(x|\theta) = \theta^x(1-\theta)^{1-x} I(x)_{x\in\{0,1\}}$$

$$S(X|\theta) = \frac{\partial \ln L(\theta|X)}{\partial\theta} = \frac{X}{\theta} - \frac{1}{1-\theta} + \frac{X}{1-\theta} = \frac{X}{\theta(1-\theta)} - \frac{1}{1-\theta}$$

$$\Rightarrow \quad Var[S(X|\theta)] = \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

and $$Var[S(\underline{X}|\theta)] \overset{\text{independent}}{=\!=} \frac{m}{\theta(1-\theta)}$$

$$E[\delta^*(\underline{X}_m)] = \theta(1-\theta) = \tau(\theta) \text{ and } \tau'(\theta) = 1-2\theta$$

Hence $$Var[\delta^*(\underline{X}_m)] \geq \frac{(1-2\theta)^2}{\frac{m}{\theta(1-\theta)}} = \frac{\theta(1-\theta)(1-2\theta)^2}{m}, \quad \theta \neq \frac{1}{2}$$

f) The maximum likelihood estimator is (under certain regularity conditions) consistent.

Hence $\hat{\delta}(\underline{X}_n) \xrightarrow{P} \delta(\theta)$

$$\delta^*(\underline{X}_n) - \hat{\delta}(\underline{X}_n) = \frac{n}{n-1}\hat{\delta}(\underline{X}_n) - \hat{\delta}(\underline{X}_n) = \frac{1}{n-1}\hat{\delta}(\underline{X}_n)$$

$\hat{\delta}(\underline{X}_n) \xrightarrow{P} \delta(\theta)$ and $\frac{1}{n-1} \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow \frac{1}{n-1}\hat{\delta}(\underline{X}_n) \xrightarrow{P} 0.$

Hence $\forall \varepsilon > 0$, $P(|\delta^*(\underline{X}_n) - \hat{\delta}(\underline{X}_n)| < \varepsilon) = P(|\frac{1}{n-1}\hat{\delta}(\underline{X}_n)| < \varepsilon) \underset{n\to\infty}{\longrightarrow} 1.$

g) Convergence in probability implies convergence in distribution. The asymptotic distribution of $\sqrt{n}(\delta^*(X_n) - \delta(\theta))$ is equal to the asymptotic distribution of $\sqrt{n}(\hat{\delta}(\underline{X}_n) - \delta(\theta))$.

Let $\frac{Y_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}$. We know $\sqrt{n}\left(\frac{Y_n}{n} - \theta\right) \xrightarrow{D} N(0, \theta(1-\theta))$ from the central limit theorem.

Let $g\left(\frac{Y_n}{n}\right) = \frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right)$, $g'(\theta) = 1 - 2\theta \neq 0$ for $\theta \neq \frac{1}{2}$.

$$\Rightarrow \sqrt{n}\left(\frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) - \theta(1-\theta)\right) \xrightarrow{D} N(0, \theta(1-\theta)(g'(\theta))^2) = N(0, \theta(1-\theta)(1-2\theta)^2),$$

Since the asymptotic variance equals $\frac{(\tau'(\theta))^2}{\operatorname{Var} S(X|\theta)}$, $\delta^*(X_n)$ is asymptotic efficient. Here $\tau(\theta) = \theta(1-\theta)$.

For $\theta = \frac{1}{2}$, $g'(\theta) = 0$. Hence,

$$n\left(\frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) - \theta(1-\theta)\right) = n\left(\frac{Y_n}{n}\left(1 - \frac{Y_n}{n}\right) - \frac{1}{4}\right) \xrightarrow{D} \frac{1}{4}\,\frac{g''(\theta)}{2}\,\chi^2(1)$$

$$= -\frac{1}{4}\chi^2(1)$$

Problem 2

a) $Y \sim \Gamma(\alpha,\beta) \Rightarrow f(y|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-\frac{y}{\beta}}, \quad y>0, \alpha>0, \beta>0$

Hence $X_1, \dots, X_m$ are all $\Gamma(2, \frac{1}{\lambda})$

$$f(x_1,\dots,x_m|\lambda) = \lambda^{2m} \prod_{i=1}^{m} x_i \, e^{-\lambda \sum_{i=1}^{m} x_i}$$

$$\Rightarrow \ln L(\lambda | x_1,\dots,x_m) = 2m \ln\lambda + \sum_{i=1}^{m} \ln x_i - \lambda \sum_{i=1}^{m} x_i$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{2m}{\lambda} - \sum_{i=1}^{m} x_i = 0 \Rightarrow \lambda = \frac{2m}{\sum_{i=1}^{m} x_i}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{2m}{\lambda^2} < 0 \Rightarrow \text{MLE of } \lambda, \ \hat{\lambda} = \frac{2m}{\sum_{i=1}^{m} X_i}$$

b) For the gamma distribution $E[X^m] = \frac{\Gamma(\alpha+m)\beta^m}{\Gamma(\alpha)}$

$$Y = \sum_{i=1}^{m} X_i \sim \Gamma(2m, \tfrac{1}{\lambda})$$

and $E[Y^{-1}] = \frac{\Gamma(\alpha-1)(\frac{1}{\lambda})^{-1}}{\Gamma(\alpha)} = \frac{\lambda}{\alpha-1} = \frac{\lambda}{2m-1}$

$$E[Y^{-2}] = \frac{\Gamma(\alpha-2)(\frac{1}{\lambda})^{-2}}{\Gamma(\alpha)} = \frac{\lambda^2}{(\alpha-1)(\alpha-2)} = \frac{\lambda^2}{(2m-1)(2m-2)}$$

Thereby $E[\hat{\lambda}] = \frac{2m\lambda}{2m-1}$

$$Var[\hat{\lambda}] = 4m^2\left[\frac{\lambda^2}{(2m-1)(2m-2)} - \frac{\lambda^2}{(2m-1)^2}\right] = \frac{4m^2\lambda^2}{2m-1}\left[\frac{1}{2m-2} - \frac{1}{2m-1}\right]$$

$$= \frac{4m^2\lambda^2}{(2m-1)^2(2m-2)}$$

$$2\lambda \sum_{i=1}^{m} X_i \sim \Gamma(2m, 2) \sim \chi^2(4m)$$

$$\Rightarrow \quad P\left(\chi^2(4m)_{1-\frac{\alpha}{2}} \le 2\lambda \sum_{i=1}^{m} X_i \le \chi^2(4m)_{\frac{\alpha}{2}} \right) = 1-\alpha$$

$\Rightarrow$ that a $(1-\alpha)$ confidence interval is given by

$$\left[\frac{\chi^2(4m)_{1-\frac{\alpha}{2}}}{2\sum_{i=1}^{m} X_i}, \frac{\chi^2(4m)_{\frac{\alpha}{2}}}{2\sum_{i=1}^{m} X_i} \right]$$

c)

$$f(x,\lambda) = \lambda^{2m} \prod_{i=1}^{m} x_i \, e^{-\lambda \sum_{i=1}^{m} x_i} \cdot \theta e^{-\theta\lambda} = \lambda^{2m} e^{-\lambda(\theta + \sum_{i=1}^{m} x_i)} \prod_{i=1}^{m} x_i \cdot \theta$$

$$= \underbrace{\frac{1}{\Gamma(2m+1)\left(\frac{1}{\theta + \sum_1^m x_i}\right)^{m+1}} \cdot \lambda^{2m} e^{-(\theta + \sum_1^m x_i)\lambda}}_{\Gamma\left(2m+1, \frac{1}{\theta + \sum_1^m x_i}\right)} \cdot \prod_{i=1}^{m} x_i \cdot \theta \cdot \frac{\Gamma(2m+1)}{(\theta + \sum_1^m x_i)^{m+1}}$$

Hence $\pi(\lambda \mid x_1, \ldots, x_m) \sim \Gamma\left(2m+1, \frac{1}{\theta + \sum_1^m x_i}\right)$

Thereby $\hat{\lambda}_B = \dfrac{2m+1}{\theta + \sum_1^m x_i}$

In the aposteriori distribution $2(\theta + \sum_1^m x_i)\lambda \sim \chi^2(4m+2)$

Therefore $P\left(\chi^2(4m+2)_{1-\frac{\alpha}{2}} \le 2(\theta + \sum_1^m x_i)\lambda \le \chi^2(4m+2)_{\frac{\alpha}{2}} \right) = 1-\alpha$

and a $1-\alpha$ credible interval is.

$$\left[\frac{\chi^2(4m+2)_{1-\frac{\alpha}{2}}}{2(\theta + \sum_1^m x_i)}, \frac{\chi^2(4m+2)_{\frac{\alpha}{2}}}{2(\theta + \sum_1^m x_i)} \right]$$